Chapter 2
Quantisation of the Electromagnetic Field

Abstract The study of the quantum features of light requires the quantisation of the electromagnetic field. In this chapter we quantise the field and introduce three possible sets of basis states, namely, the Fock or number states, the coherent states and the squeezed states. The properties of these states are discussed. The phase operator and the associated phase states are also introduced.

2.1 Field Quantisation

The major emphasis of this text is concerned with the uniquely quantum-mechanical properties of the electromagnetic field, which are not present in a classical treatment. As such we shall begin immediately by quantizing the electromagnetic field. We shall make use of an expansion of the vector potential for the electromagnetic field in terms of cavity modes. The problem then reduces to the quantization of the harmonic oscillator corresponding to each individual cavity mode.

We shall also introduce states of the electromagnetic field appropriate to the description of optical fields. The first set of states we introduce are the number states corresponding to having a definite number of photons in the field. It turns out that it is extremely difficult to create experimentally a number state of the field, though fields containing a very small number of photons have been generated. A more typical optical field will involve a superposition of number states. One such field is the coherent state of the field which has the minimum uncertainty in amplitude and phase allowed by the uncertainty principle, and hence is the closest possible quantum mechanical state to a classical field. It also possesses a high degree of optical coherence as will be discussed in Chap. 3, hence the name coherent state. The coherent state plays a fundamental role in quantum optics and has a practical significance in that a highly stabilized laser operating well above threshold generates a coherent state.

A rather more exotic set of states of the electromagnetic field are the squeezed states. These are also minimum-uncertainty states but unlike the coherent states the quantum noise is not uniformly distributed in phase. Squeezed states may have less noise in one quadrature than the vacuum. As a consequence the noise in the other quadrature is increased. We introduce the basic properties of squeezed states in this chapter. In Chap. 8 we describe ways to generate squeezed states and their applications.

While states of definite photon number are readily defined as eigenstates of the number operator a corresponding description of states of definite phase is more difficult. This is due to the problems involved in constructing a Hermitian phase operator to describe a bounded physical quantity like phase. How this problem may be resolved together with the properties of phase states is discussed in the final section of this chapter.

A convenient starting point for the quantisation of the electromagnetic field is the classical field equations. The free electromagnetic field obeys the source free Maxwell equations.

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{2.1a}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{2.1b}$$

$$\nabla \cdot \boldsymbol{D} = 0\,, \tag{2.1c}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{2.1d}$$

where $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, $D = \varepsilon_0 \boldsymbol{E}$, μ_0 and ε_0 being the magnetic permeability and electric permittivity of free space, and $\mu_0\varepsilon_0 = c^{-2}$. Maxwell's equations are gauge invariant when no sources are present. A convenient choice of gauge for problems in quantum optics is the Coulomb gauge. In the Coulomb gauge both $\boldsymbol{B}$ and $\boldsymbol{E}$ may be determined from a vector potential $\boldsymbol{A}(\boldsymbol{r},\, t)$ as follows

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}\,, \tag{2.2a}$$

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}\,, \tag{2.2b}$$

with the Coulomb gauge condition

$$\nabla \cdot \boldsymbol{A} = 0\,. \tag{2.3}$$

Substituting (2.2a) into (2.1d) we find that $\boldsymbol{A}(\boldsymbol{r},\, t)$ satisfies the wave equation

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2}\,. \tag{2.4}$$

We separate the vector potential into two complex terms

$$\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}^{(+)}(\boldsymbol{r},t) + \boldsymbol{A}^{(-)}(\boldsymbol{r},t)\,, \tag{2.5}$$

where $\boldsymbol{A}^{(+)}(\boldsymbol{r},\, t)$ contains all amplitudes which vary as $\mathrm{e}^{-\mathrm{i}\omega t}$ for $\omega > 0$ and $\boldsymbol{A}^{(-)}(\boldsymbol{r},\, t)$ contains all amplitudes which vary as $\mathrm{e}^{\mathrm{i}\omega t}$ and $\boldsymbol{A}^{(-)} = (\boldsymbol{A}^{(+)})^*$.

It is more convenient to deal with a discrete set of variables rather than the whole continuum. We shall therefore describe the field restricted to a certain volume of space and expand the vector potential in terms of a discrete set of orthogonal mode functions:

$$\boldsymbol{A}^{(+)}(\boldsymbol{r},t)=\sum_k c_k \boldsymbol{u}_k(\boldsymbol{r})e^{-i\omega_k t}\,, \tag{2.6}$$

where the Fourier coefficients c_k are constant for a free field. The set of vector mode functions $\boldsymbol{u}_k(\boldsymbol{r})$ which correspond to the frequency ω_k will satisfy the wave equation

$$\left(\nabla^2+\frac{\omega_k^2}{c^2}\right)\boldsymbol{u}_k(\boldsymbol{r})=0 \tag{2.7}$$

provided the volume contains no refracting material. The mode functions are also required to satisfy the transversality condition,

$$\nabla\cdot\boldsymbol{u}_k(\boldsymbol{r})=0\,. \tag{2.8}$$

The mode functions form a complete orthonormal set

$$\int_V \boldsymbol{u}_k^*(\boldsymbol{r})\,\boldsymbol{u}_{k'}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}=\delta_{kk'}\,. \tag{2.9}$$

The mode functions depend on the boundary conditions of the physical volume under consideration, e.g., periodic boundary conditions corresponding to travelling-wave modes or conditions appropriate to reflecting walls which lead to standing waves. For example, the plane wave mode functions appropriate to a cubical volume of side L may be written as

$$\boldsymbol{u}_k(\boldsymbol{r})=L^{-3/2}\hat{\boldsymbol{e}}^{(\lambda)}\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) \tag{2.10}$$

where $\hat{\boldsymbol{e}}^{(\lambda)}$ is the unit polarization vector. The mode index k describes several discrete variables, the polarisation index $(\lambda=1,\ 2)$ and the three Cartesian components of the propagation vector $\boldsymbol{k}$. Each component of the wave vector $\boldsymbol{k}$ takes the values

$$k_x=\frac{2\pi n_x}{L},\quad k_y=\frac{2\pi n_y}{L},\quad k_z=\frac{2\pi n_z}{L},\qquad n_x,n_y,n_z=0,\pm1,\pm2,\ldots \tag{2.11}$$

The polarization vector $\hat{\boldsymbol{e}}^{(\lambda)}$ is required to be perpendicular to $\boldsymbol{k}$ by the transversality condition (2.8).

The vector potential may now be written in the form

$$\boldsymbol{A}(\boldsymbol{r},t)=\sum_k\left(\frac{\hbar}{2\omega_k\varepsilon_0}.\right)^{1/2}\left[a_k\boldsymbol{u}_k(\boldsymbol{r})\mathrm{e}^{-i\omega_k t}+a_k^\dagger\boldsymbol{u}_k^*(\boldsymbol{r})\,\mathrm{e}^{i\omega_k t}\right]\,. \tag{2.12}$$

The corresponding form for the electric field is

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{i}\sum_k \left(\frac{\hbar\omega_k}{2\varepsilon_0}\right)^{1/2} \left[a_k \boldsymbol{u}_k(\boldsymbol{r})\,\mathrm{e}^{-i\omega_k t} - a_k^\dagger \boldsymbol{u}_k^*(\boldsymbol{r})\mathrm{e}^{i\omega_k t}\right] . \tag{2.13}$$

The normalization factors have been chosen such that the amplitudes a_k and $a_k^\dagger$ are dimensionless.

In classical electromagnetic theory these Fourier amplitudes are complex numbers. Quantisation of the electromagnetic field is accomplished by choosing a_k and $a_k^\dagger$ to be mutually adjoint operators. Since photons are bosons the appropriate commutation relations to choose for the operators a_k and $a_k^\dagger$ are the boson commutation relations

$$[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0, \quad \left[a_k, a_{k'}^\dagger\right] = \delta_{kk'} . \tag{2.14}$$

The dynamical behaviour of the electric-field amplitudes may then be described by an ensemble of independent harmonic oscillators obeying the above commutation relations. The quantum states of each mode may now be discussed independently of one another. The state in each mode may be described by a state vector $|\Psi\rangle_k$ of the Hilbert space appropriate to that mode. The states of the entire field are then defined in the tensor product space of the Hilbert spaces for all of the modes.

The Hamiltonian for the electromagnetic field is given by

$$H = \frac{1}{2}\int \left(\varepsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2\right) \mathrm{d}\boldsymbol{r} . \tag{2.15}$$

Substituting (2.13) for $\boldsymbol{E}$ and the equivalent expression for $\boldsymbol{H}$ and making use of the conditions (2.8) and (2.9), the Hamiltonian may be reduced to the form

$$H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right) . \tag{2.16}$$

This represents the sum of the number of photons in each mode multiplied by the energy of a photon in that mode, plus $\frac{1}{2}\hbar\omega_k$ representing the energy of the vacuum fluctuations in each mode. We shall now consider three possible representations of the electromagnetic field.

2.2 Fock or Number States

The Hamiltonian (2.15) has the eigenvalues $h\omega_k(n_k + \frac{1}{2})$ where n_k is an integer $(n_k = 0, 1, 2, \ldots, \infty)$. The eigenstates are written as $|n_k\rangle$ and are known as number or Fock states. They are eigenstates of the number operator $N_k = a_k^\dagger a_k$

$$a_k^\dagger a_k |n_k\rangle = n_k |n_k\rangle . \tag{2.17}$$

The ground state of the oscillator (or vacuum state of the field mode) is defined by

$$a_k|0\rangle = 0\,. \tag{2.18}$$

From (2.16 and 2.18) we see that the energy of the ground state is given by

$$\langle 0|H|0 = \frac{1}{2}\sum_k \hbar\omega_k\,. \tag{2.19}$$

Since there is no upper bound to the frequencies in the sum over electromagnetic field modes, the energy of the ground state is infinite, a conceptual difficulty of quantized radiation field theory. However, since practical experiments measure a change in the total energy of the electromagnetic field the infinite zero-point energy does not lead to any divergence in practice. Further discussions on this point may be found in [1]. a_k and $a_k^\dagger$ are raising and lowering operators for the harmonic oscillator ladder of eigenstates. In terms of photons they represent the annihilation and creation of a photon with the wave vector $\boldsymbol{k}$ and a polarisation $\hat{\boldsymbol{e}}_k$. Hence the terminology, annihilation and creation operators. Application of the creation and annihilation operators to the number states yield

$$a_k|n_k\rangle = n_k^{1/2}|n_k-1\rangle, \quad a_k^\dagger|n_k\rangle = (n_k+1)^{1/2}|n_k+1\rangle\,. \tag{2.20}$$

The state vectors for the higher excited states may be obtained from the vacuum by successive application of the creation operator

$$|n_k\rangle = \frac{\left(a_k^\dagger\right)^{n_k}}{(n_k!)^{1/2}}|0\rangle, \quad n_k = 0,1,2\ldots\,. \tag{2.21}$$

The number states are orthogonal

$$\langle n_k|m_k\rangle = \delta_{mn}\,, \tag{2.22}$$

and complete

$$\sum_{n_k=0}^{\infty}|n_k\rangle\langle n_k| = 1\,. \tag{2.23}$$

Since the norm of these eigenvectors is finite, they form a complete set of basis vectors for a Hilbert space.

While the number states form a useful representation for high-energy photons, e.g. γ rays where the number of photons is very small, they are not the most suitable representation for optical fields where the total number of photons is large. Experimental difficulties have prevented the generation of photon number states with more than a small number of photons (but see 16.4.2). Most optical fields are either a superposition of number states (pure state) or a mixture of number states (mixed state). Despite this the number states of the electromagnetic field have been used as a basis for several problems in quantum optics including some laser theories.

2.3 Coherent States

A more appropriate basis for many optical fields are the coherent states [2]. The coherent states have an indefinite number of photons which allows them to have a more precisely defined phase than a number state where the phase is completely random. The product of the uncertainty in amplitude and phase for a coherent state is the minimum allowed by the uncertainty principle. In this sense they are the closest quantum mechanical states to a classical description of the field. We shall outline the basic properties of the coherent states below. These states are most easily generated using the unitary displacement operator

$$D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right) , \tag{2.24}$$

where α is an arbitrary complex number.

Using the operator theorem [2]

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-[A,B]/2} , \tag{2.25}$$

which holds when

$$[A,[A,B]] = [B,[A,B]] = 0,$$

we can write $D(\alpha)$ as

$$D(\alpha) = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a} . \tag{2.26}$$

The displacement operator $D(\alpha)$ has the following properties

$$\begin{aligned} &D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha), \qquad D^\dagger(\alpha)\, a D(\alpha) = a + \alpha, \\ &D^\dagger(\alpha)\, a^\dagger D(\alpha) = a^\dagger + \alpha^* . \end{aligned} \tag{2.27}$$

The coherent state $|\alpha\rangle$ is generated by operating with $D(\alpha)$ on the vacuum state

$$|\alpha\rangle = D(\alpha)|0\rangle . \tag{2.28}$$

The coherent states are eigenstates of the annihilation operator a. This may be proved as follows:

$$D^\dagger(\alpha)\, a|\alpha\rangle = D^\dagger(\alpha)\, aD(\alpha)|0\rangle = (a+\alpha)|0\rangle = \alpha|0\rangle . \tag{2.29}$$

Multiplying both sides by $D(\alpha)$ we arrive at the eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle . \tag{2.30}$$

Since a is a non-Hermitian operator its eigenvalues α are complex.

Another useful property which follows using (2.25) is

$$D(\alpha+\beta) = D(\alpha)D(\beta)\exp\left(-\mathrm{i}\,\mathrm{Im}\{\alpha\beta^*\}\right) . \tag{2.31}$$

The coherent states contain an indefinite number of photons. This may be made apparent by considering an expansion of the coherent states in the number states basis.

Taking the scalar product of both sides of (2.30) with $\langle n|$ we find the recursion relation

$$(n+1)^{1/2}\langle n+1|\alpha\rangle = \alpha\langle n|\alpha\rangle\ . \tag{2.32}$$

It follows that

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{1/2}}\langle 0|\alpha\rangle\ . \tag{2.33}$$

We may expand $|\alpha\rangle$ in terms of the number states $|n\rangle$ with expansion coefficients $\langle n|\alpha\rangle$ as follows

$$|\alpha\rangle = \sum |n\rangle\langle n|\alpha\rangle = \langle 0|\alpha\rangle \sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle\ . \tag{2.34}$$

The squared length of the vector $|\alpha\rangle$ is thus

$$|\langle\alpha|\alpha\rangle|^2 = |\langle 0|\alpha\rangle|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 \mathrm{e}^{|\alpha|^2}\ . \tag{2.35}$$

It is easily seen that

$$\begin{aligned}\langle 0|\alpha\rangle &= \langle 0|D(\alpha)|0\rangle \\ &= \mathrm{e}^{-|\alpha|^2/2}\ .\end{aligned} \tag{2.36}$$

Thus $|\langle\alpha|\alpha\rangle|^2 = 1$ and the coherent states are normalized.

The coherent state may then be expanded in terms of the number states as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum \frac{\alpha^n}{(n!)^{1/2}}|n\rangle\ . \tag{2.37}$$

We note that the probability distribution of photons in a coherent state is a Poisson distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}\mathrm{e}^{-|\alpha|^2}}{n!}\ , \tag{2.38}$$

where $|\alpha|^2$ is the mean number of photons ($\bar{n} = \langle\alpha|a^\dagger\, a|\alpha\rangle = |\alpha|^2$).

The scalar product of two coherent states is

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(\beta)D(\alpha)|0\rangle\ . \tag{2.39}$$

Using (2.26) this becomes

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) + \alpha\beta^*\right]\ . \tag{2.40}$$

The absolute magnitude of the scalar product is

$$|\langle\beta|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2} . \tag{2.41}$$

Thus the coherent states are not orthogonal although two states $|\alpha\rangle$ and $|\beta\rangle$ become approximately orthogonal in the limit $|\alpha-\beta| \gg 1$. The coherent states form a two-dimensional continuum of states and are, in fact, overcomplete. The completeness relation

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha = 1 , \tag{2.42}$$

may be proved as follows.

We use the expansion (2.37) to give

$$\int |\alpha\rangle\langle\alpha| \frac{\mathrm{d}^2\alpha}{\pi} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\pi\sqrt{n!m!}} \int \mathrm{e}^{-|\alpha|^2}\alpha^{*m}\alpha^n \mathrm{d}^2\alpha . \tag{2.43}$$

Changing to polar coordinates this becomes

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = \sum_{n,m=0}^{\infty} \frac{|n\rangle\langle m|}{\pi\sqrt{n!m!}} \int_0^{\infty} r\mathrm{d}r\mathrm{e}^{-r^2} r^{n+m} \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\mathrm{i}(n-m)\theta} . \tag{2.44}$$

Using

$$\int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{\mathrm{i}(n-m)\theta} = 2\pi\delta_{nm} , \tag{2.45}$$

we have

$$\int |\alpha\rangle\langle\alpha| \frac{\mathrm{d}^2\alpha}{\pi} = \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^{\infty} \mathrm{d}\varepsilon\, \mathrm{e}^{-\varepsilon}\varepsilon^n , \tag{2.46}$$

where we let $\varepsilon = r^2$. The integral equals $n!$. Hence we have

$$\int |\alpha\rangle\langle\alpha| \frac{\mathrm{d}^2\alpha}{\pi} = \sum_{n=0}^{\infty} |n\rangle\langle n = 1 , \tag{2.47}$$

following from the completeness relation for the number states.

An alternative proof of the completeness of the coherent states may be given as follows. Using the relation [3]

$$\mathrm{e}^{\zeta B} A \mathrm{e}^{-\zeta B} = A + \zeta[B,A] + \frac{\zeta^2}{2!}[B,[B,A]] + \cdots , \tag{2.48}$$

it is easy to see that all the operators A such that

$$D^{\dagger}(\alpha) A D(\alpha) = A \tag{2.49}$$

are proportional to the identity.

We consider

$$A = \int d^2\alpha |\alpha\rangle\langle\alpha|$$

then

$$D^\dagger(\beta) \int d^2\alpha |\alpha\rangle\langle\alpha| D(\beta) = \int d^2\alpha |\alpha - \beta\rangle\langle\alpha - \beta| = \int d^2\alpha |\alpha\rangle\langle\alpha| \,. \tag{2.50}$$

Then using the above result we conclude that

$$\int d^2\alpha |\alpha\rangle\langle\alpha| \propto I \,. \tag{2.51}$$

The constant of proportionality is easily seen to be π.

The coherent states have a physical significance in that the field generated by a highly stabilized laser operating well above threshold is a coherent state. They form a useful basis for expanding the optical field in problems in laser physics and nonlinear optics. The coherence properties of light fields and the significance of the coherent states will be discussed in Chap. 3.

2.4 Squeezed States

A general class of minimum-uncertainty states are known as *squeezed states*. In general, a squeezed state may have less noise in one quadrature than a coherent state. To satisfy the requirements of a minimum-uncertainty state the noise in the other quadrature is greater than that of a coherent state. The coherent states are a particular member of this more general class of minimum uncertainty states with equal noise in both quadratures. We shall begin our discussion by defining a family of minimum-uncertainty states. Let us calculate the variances for the position and momentum operators for the harmonic oscillator

$$q = \sqrt{\frac{\hbar}{2\omega}}\left(a + a^\dagger\right), \qquad p = \mathrm{i}\sqrt{\frac{\hbar\omega}{2}}\left(a - a^\dagger\right) \,. \tag{2.52}$$

The variances are defined by

$$V(A) = (\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 \,. \tag{2.53}$$

In a coherent state we obtain

$$(\Delta q)^2_{\mathrm{coh}} = \frac{\hbar}{2\omega}, \quad (\Delta p)^2_{\mathrm{coh}} = \frac{\hbar\omega}{2} \,. \tag{2.54}$$

Thus the product of the uncertainties is a minimum

$$(\Delta p \Delta q)_{\mathrm{coh}} = \frac{\hbar}{2} \,. \tag{2.55}$$

Thus, there exists a sense in which the description of the state of an oscillator by a coherent state represents as close an approach to classical localisation as possible. We shall consider the properties of a single-mode field. We may write the annihilation operator a as a linear combination of two Hermitian operators

$$a = \frac{X_1 + \mathrm{i}X_2}{2} \,. \tag{2.56}$$

X_1 and X_2, the real and imaginary parts of the complex amplitude, give dimensionless amplitudes for the modes' two quadrature phases. They obey the following commutation relation

$$[X_1, X_2] = 2\mathrm{i} \tag{2.57}$$

The corresponding uncertainty principle is

$$\Delta X_1 \, \Delta X_2 \geq 1 \,. \tag{2.58}$$

This relation with the equals sign defines a family of minimum-uncertainty states. The coherent states are a particular minimum-uncertainty state with

$$\Delta X_1 = \Delta X_2 = 1 \,. \tag{2.59}$$

The coherent state $|\alpha\rangle$ has the mean complex amplitude α and it is a minimum-uncertainty state for X_1 and X_2, with equal uncertainties in the two quadrature phases. A coherent state may be represented by an "error circle" in a complex amplitude plane whose axes are X_1 and X_2 (Fig. 2.1a). The center of the error circle lies at $\frac{1}{2}\langle X_1 + \mathrm{i}X_2\rangle = \alpha$ and the radius $\Delta X_1 = \Delta X_2 = 1$ accounts for the uncertainties in X_1 and X_2.

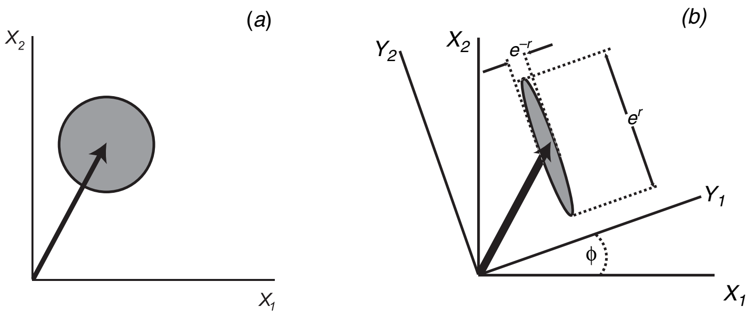

Fig. 2.1 Phase space representation showing contours of constant uncertainty for (**a**) coherent state and (**b**) squeezed state $|\alpha, \varepsilon\rangle$

There is obviously a whole family of minimum-uncertainty states defined by $\Delta X_1 \Delta X_2 = 1$. If we plot ΔX_1 against ΔX_2 the minimum-uncertainty states lie on a hyperbola (Fig. 2.2). Only points lying to the right of this hyperbola correspond to physical states. The coherent state with $\Delta X_1 = \Delta X_2$ is a special case of a more general class of states which may have reduced uncertainty in one quadrature at the expense of increased uncertainty in the other $(\Delta X_1 < 1 < \Delta X_2)$. These states correspond to the shaded region in Fig. 2.2. Such states we shall call *squeezed states* [4]. They may be generated by using the unitary squeeze operator [5]

$$S(\varepsilon) = \exp\left(1/2\varepsilon^* a^2 - 1/2\varepsilon a^{\dagger 2}\right) . \tag{2.60}$$

where $\varepsilon = r\mathrm{e}^{2\mathrm{i}\phi}$.

Note the squeeze operator obeys the relations

$$S^\dagger(\varepsilon) = S^{-1}(\varepsilon) = S(-\varepsilon) , \tag{2.61}$$

and has the following useful transformation properties

$$\begin{aligned} S^\dagger(\varepsilon)\, a S(\varepsilon) &= a\cosh r - a^\dagger \mathrm{e}^{-2\mathrm{i}\phi}\sinh r, \\ S^\dagger(\varepsilon)\, a^\dagger S(\varepsilon) &= a^\dagger \cosh r - a\mathrm{e}^{-2\mathrm{i}\phi}\sinh r , \\ S^\dagger(\varepsilon)\,(Y_1 + \mathrm{i}Y_2) S(\varepsilon) &= Y_1 \mathrm{e}^{-r} + \mathrm{i}Y_2 \mathrm{e}^{r}, \end{aligned} \tag{2.62}$$

where

$$Y_1 + \mathrm{i}Y_2 = (X_1 + \mathrm{i}X_2)\,\mathrm{e}^{-\mathrm{i}\phi} \tag{2.63}$$

is a rotated complex amplitude. The squeeze operator attenuates one component of the (rotated) complex amplitude, and it amplifies the other component. The degree of attenuation and amplification is determined by $r = |\varepsilon|$, which will be called the *squeeze factor*. The squeezed state $|\alpha, \varepsilon\rangle$ is obtained by first squeezing the vacuum and then displacing it

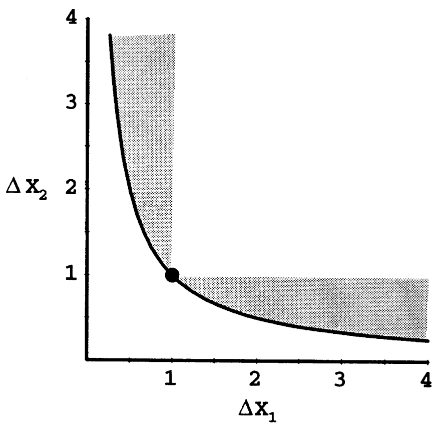

Fig. 2.2 Plot of ΔX_1 versus ΔX_2 for the minimum-uncertainty states. The *dot* marks a coherent state while the *shaded region* corresponds to the squeezed states

$$|\alpha,\varepsilon\rangle = D(\alpha)\,S(\varepsilon)\,|0\rangle\ . \tag{2.64}$$

A squeezed state has the following expectation values and variances

$$\begin{aligned}
&\langle X_1 + \mathrm{i}X_2\rangle = \langle Y_1 + \mathrm{i}Y_2\rangle \mathrm{e}^{\mathrm{i}\phi} = 2\alpha,\\
&\Delta Y_1 = \mathrm{e}^{-r}, \qquad \Delta Y_2 = \mathrm{e}^{r},\\
&\langle N\rangle = |\alpha^2| + \sinh^2 r,\\
&(\Delta N)^2 = |\alpha \cosh r - \alpha^* \mathrm{e}^{2\mathrm{i}\phi} \sinh r|^2 + 2\cosh^2 r \sinh^2 r\ .
\end{aligned} \tag{2.65}$$

Thus the squeezed state has unequal uncertainties for Y_1 and Y_2 as seen in the error ellipse shown in Fig. 2.1b. The principal axes of the ellipse lie along the Y_1 and Y_2 axes, and the principal radii are ΔY_1 and ΔY_2. A more rigorous definition of these error ellipses as contours of the Wigner function is given in Chap. 3.

2.5 Two-Photon Coherent States

We may define squeezed states in an alternative but equivalent way [6]. As this definition is sometimes used in the literature we include it for completeness.

Consider the operator

$$b = \mu a + \nu a^\dagger \tag{2.66}$$

where

$$|\mu|^2 - |\nu|^2 = 1\ .$$

Then b obeys the commutation relation

$$\left[b, b^\dagger\right] = 1\ . \tag{2.67}$$

We may write (2.66) as

$$b = U a U^\dagger \tag{2.68}$$

where U is a unitary operator. The eigenstates of b have been called *two-photon coherent states* and are closely related to the squeezed states.

The eigenvalue equation may be written as

$$b|\beta\rangle_{\mathrm{g}} = \beta|\beta\rangle_{\mathrm{g}}\ . \tag{2.69}$$

From (2.68) it follows that

$$|\beta\rangle_{\mathrm{g}} = U|\beta\rangle \tag{2.70}$$

where $|\beta\rangle$ are the eigenstates of a.

The properties of $|\beta\rangle_{\mathrm{g}}$ may be proved to parallel those of the coherent states. The state $|\beta\rangle_{\mathrm{g}}$ may be obtained by operating on the vacuum

$$|\beta\rangle_{\mathrm{g}} = D_{\mathrm{g}}(\beta)\,|0\rangle_{\mathrm{g}} \tag{2.71}$$

with the displacement operator

$$D_g(\beta) = e^{\beta b^\dagger - \beta^* b} \tag{2.72}$$

and $|0\rangle_g = U|0\rangle$. The two-photon coherent states are complete

$$\int |\beta\rangle_{g\,g}\langle\beta| \frac{d^2\beta}{\pi} = 1 \tag{2.73}$$

and their scalar product is

$${}_g\langle\beta|\beta'\rangle_g = \exp\left(\beta^*\beta' - \frac{1}{2}|\beta|^2 - \frac{1}{2}|\beta'|^2\right) . \tag{2.74}$$

We now consider the relation between the two-photon coherent states and the squeezed states as previously defined. We first note that

$$U \equiv S(\varepsilon)$$

with $\mu = \cosh r$ and $\nu = e^{2i\phi}\sinh r$. Thus

$$|0\rangle_g \equiv |0, \varepsilon\rangle \tag{2.75}$$

with the above relations between (μ, ν) and $(r,\ \theta)$. Using this result in (2.71) and rewriting the displacement operator, $D_g(\beta)$, in terms of a and $a^\dagger$ we find

$$|\beta\rangle_g = D(\alpha)S(\varepsilon)|0\rangle = |\alpha, \varepsilon| \tag{2.76}$$

where

$$\alpha = \mu\beta - \nu\beta^* .$$

Thus we have found the equivalent squeezed state for the given two-photon coherent state.

Finally, we note that the two-photon coherent state $|\beta\rangle_g$ may be written as

$$|\beta\rangle_g = S(\varepsilon)D(\beta)|0\rangle .$$

Thus the two-photon coherent state is generated by first displacing the vacuum state, then squeezing. This is the opposite procedure to that which defines the squeezed state $|\alpha,\ \varepsilon\rangle$. The two procedures yield the same state if the displacement parameters α and β are related as discussed above.

The completeness relation for the two-photon coherent states may be employed to derive the completeness relation for the squeezed states. Using the above results we have

$$\int \frac{d^2\beta}{\pi} |\beta\cosh r - \beta^* e^{2i\phi}\sinh r,\ \varepsilon\rangle\langle\beta\cosh r - \beta^* e^{2i\phi}\sinh r,\ \varepsilon| = 1 . \tag{2.77}$$

The change of variable

$$\alpha = \beta \cosh r - \beta^* e^{2i\phi} \sinh r \tag{2.78}$$

leaves the measure invariant, that is $d^2\alpha = d^2\beta$. Thus

$$\int \frac{d^2\alpha}{\pi} |\alpha,\ \varepsilon\rangle\langle\alpha, \varepsilon| = 1 \,. \tag{2.79}$$

2.6 Variance in the Electric Field

The electric field for a single mode may be written in terms of the operators X_1 and X_2 as

$$E(\boldsymbol{r},t) = \frac{1}{\sqrt{L^3}} \left(\frac{\hbar\omega}{2\varepsilon_0}\right)^{1/2} \left[X_1 \sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) - X_2 \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\right] . \tag{2.80}$$

The variance in the electric field is given by

$$\begin{aligned} V(E(\boldsymbol{r},t)) = K\big\{&V(X_1)\sin^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) + V(X_2)\cos^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) \\ &- \sin[2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})]\, V(X_1, X_2)\big\} \end{aligned} \tag{2.81}$$

where

$$K = \frac{1}{L^3}\left(\frac{2\hbar\omega}{\varepsilon_0}\right),$$

$$V(X_1, X_2) = \frac{\langle (X_1X_2) + (X_2X_1)\rangle}{2} - \langle X_1\rangle\langle X_2\rangle.$$

For a minimum-uncertainty state

$$V(X_1, X_2) = 0 \,. \tag{2.82}$$

Hence (2.81) reduces to

$$V(E(\boldsymbol{r},t)) = K\left[V(X_1)\sin^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) + V(X_2)\cos^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\right] . \tag{2.83}$$

The mean and uncertainty of the electric field is exhibited in Figs. 2.3a–c where the line is thickened about a mean sinusoidal curve to represent the uncertainty in the electric field.

The variance of the electric field for a coherent state is a constant with time (Fig. 2.3a). This is due to the fact that while the coherent-state-error circle rotates about the origin at frequency ω, it has a constant projection on the axis defining the electric field. Whereas for a squeezed state the rotation of the error ellipse leads to a variance that oscillates with frequency 2ω. In Fig. 2.3b the coherent excitation

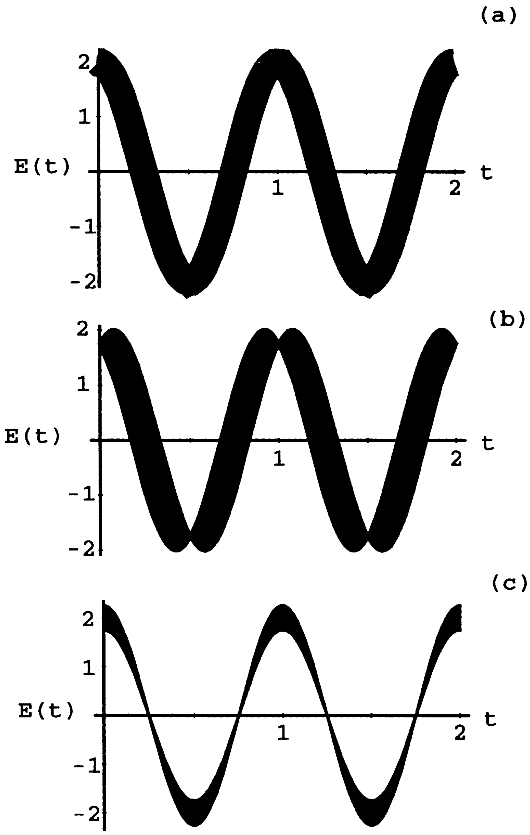

Fig. 2.3 Plot of the electric field versus time showing schematically the uncertainty in phase and amplitude for (**a**) a coherent state, (**b**) a squeezed state with reduced amplitude fluctuations, and (**c**) a squeezed state with reduced phase fluctuations

appears in the quadrature that has reduced noise. In Fig. 2.3c the coherent excitation appears in the quadrature with increased noise. This situation corresponds to the phase states discussed in [7] and in the final section of this chapter.

The squeezed state $|\alpha,\ r\rangle$ has the photon number distribution [6]

$$p(n) = (n!\cosh r)^{-1}\left[\frac{1}{2}\tanh r\right]^{n}\exp\left[-|\alpha|^{2}-\frac{1}{2}\tanh r\left((\alpha^{*})^{2}\,\mathrm{e}^{\mathrm{i}\phi}+\alpha^{2}\mathrm{e}^{-\mathrm{i}\phi}\right)\right]|H_{n}(z)|^{2} \tag{2.84}$$

where

$$z=\frac{\alpha+\alpha^{*}\mathrm{e}^{\mathrm{i}\phi}\tanh r}{\sqrt{2\mathrm{e}^{\mathrm{i}\phi}\tanh r}}\,.$$

The photon number distribution for a squeezed state may be broader or narrower than a Poissonian depending on whether the reduced fluctuations occur in the phase (X_2) or amplitude (X_1) component of the field. This is illustrated in Fig. 2.4a where we plot $P(n)$ for $r=0,\ r>0$, and $r<0$. Note, a squeezed vacuum ($\alpha=0$) contains only even numbers of photons since $H_n(0)=0$ for n odd.

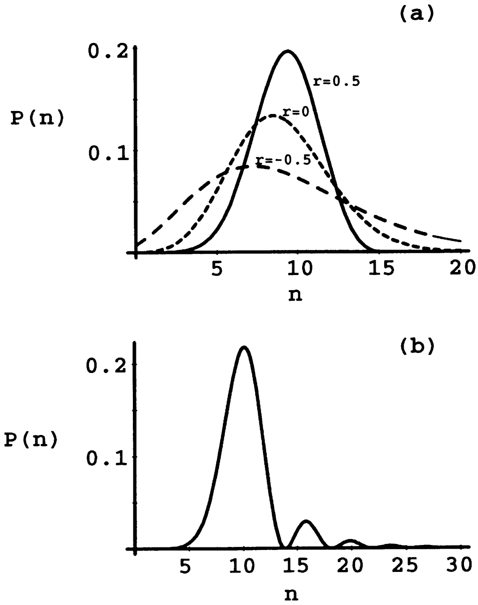

Fig. 2.4 Photon number distribution for a squeezed state $|\alpha, r\rangle$: (**a**) $\alpha = 3$, $r = 0$, 0.5, -0.5, (**b**) $\alpha = 3$, $r = 1.0$

For larger values of the squeeze parameter r, the photon number distribution exhibits oscillations, as depicted in Fig. 2.4b. These oscillations have been interpreted as interference in phase space [8].

2.7 Multimode Squeezed States

Multimode squeezed states are important since several devices produce light which is correlated at the two frequencies ω_+ and ω_-. Usually these frequencies are symmetrically placed either side of a carrier frequency. The squeezing exists not in the single modes but in the correlated state formed by the two modes.

A two-mode squeezed state may be defined by [9]

$$|\alpha_+, \alpha_-\rangle = D_+(\alpha_+) D_-(\alpha_-) S(G) |0\rangle \tag{2.85}$$

where the displacement operator is

$$D_\pm(\alpha) = \exp\left(\alpha a_\pm^\dagger - \alpha^* a_\pm\right) , \tag{2.86}$$

and the unitary two-mode squeeze operator is

$$S(G) = \exp\left(G^* a_+ a_- - G a_+^\dagger a_-^\dagger\right) . \tag{2.87}$$

The squeezing operator transforms the annihilation operators as

$$S^\dagger(G) a_\pm \, S(G) = a_\pm \cosh r - a_\mp^\dagger \, e^{i\theta} \sinh r , \tag{2.88}$$

where $G = re^{i\theta}$.

This gives for the following expectation values

$$\begin{aligned} \langle a_\pm \rangle &= \alpha_\pm \\ \langle a_\pm a_\pm \rangle &= \alpha_\pm^2 \\ \langle a_+ a_- \rangle &= \alpha_+ \alpha_- - e^{i\theta} \, \sinh r \cosh r \\ \langle a_\pm^\dagger a_\pm \rangle &= |\alpha_\pm|^2 + \sinh^2 \, r. \end{aligned} \tag{2.89}$$

The quadrature operator X is generalized in the two-mode case to

$$X = \frac{1}{\sqrt{2}} \left(a_+ + a_+^\dagger + a_- + a_-^\dagger\right) . \tag{2.90}$$

As will be seen in Chap. 5, this definition is a particular case of a more general definition. It corresponds to the degenerate situation in which the frequencies of the two modes are equal.

The mean and variance of X in a two-mode squeezed state is

$$\begin{aligned} \langle X \rangle &= 2(\mathrm{Re}\{\alpha_+\} + \mathrm{Re}\{\alpha_-\}), \\ V(X) &= \left(e^{-2r} \cos^2 \frac{\theta}{2} + e^{2r} \sin^2 \frac{\theta}{2}\right) . \end{aligned} \tag{2.91}$$

These results for two-mode squeezed states will be used in the analyses of nondegenerate parametric oscillation given in Chaps. 4 and 6.

2.8 Phase Properties of the Field

The definition of an Hermitian phase operator corresponding to the physical phase of the field has long been a problem. Initial attempts by P. Dirac led to a non-Hermitian operator with incorrect commutation relations. Many of these difficulties were made quite explicit in the work of *Susskind* and *Glogower* [10]. *Pegg* and *Barnett* [11] showed how to construct an Hermitian phase operator, the eigenstates of which, in an appropriate limit, generate the correct phase statistics for arbitrary states. We will first discuss the *Susskind–Glogower* (SG) phase operator.

Let a be the annihilation operator for a harmonic oscillator, representing a single field mode. In analogy with the classical polar decomposition of a complex amplitude we define the SG phase operator,

$$e^{i\phi} = \left(aa^{\dagger}\right)^{-1/2} a \,. \tag{2.92}$$

The operator $e^{i\phi}$ has the number state expansion

$$e^{i\phi} = \sum_{n=1}^{\infty} |n\rangle\langle n+1| \tag{2.93}$$

and eigenstates $|e^{i\phi}\rangle$ like

$$|e^{i\phi}\rangle = \sum_{n=1}^{\infty} e^{in\phi}|n\rangle \quad \text{for} \quad -\pi < \phi \le \pi \,. \tag{2.94}$$

It is easy to see from (2.93) that $e^{i\phi}$ is not unitary,

$$\left[e^{i\phi}, \left(e^{i\phi}\right)^{\dagger}\right] = |0\rangle\langle 0| \,. \tag{2.95}$$

An equivalent statement is that the SG phase operator is not Hermitian. As an immediate consequence the eigenstates $|e^{i\phi}\rangle$ are not orthogonal. In many ways this is similar to the non-orthogonal eigenstates of the annihilation operator a, i.e. the coherent states. None-the-less these states do provide a resolution of identity

$$\int_{-\pi}^{\pi} d\phi \left|e^{i\phi}\rangle\langle e^{i\phi}\right| = 2\pi \,. \tag{2.96}$$

The phase distribution over the window $-\pi < \phi \le \pi$ for any state $|\psi\rangle$ is then defined by

$$P(\phi) = \frac{1}{2\pi}|\langle e^{i\phi}|\psi\rangle|^2 \,. \tag{2.97}$$

The normalisation integral is

$$\int_{-\pi}^{\pi} P(\phi)\, d\phi = 1 \,. \tag{2.98}$$

The question arises; does this distribution correspond to the statistics of any physical phase measurement? At the present time there does not appear to be an answer. However, there are theoretical grounds [12] for believing that $P(\phi)$ is the correct distribution for optimal phase measurements. If this is accepted then the fact that the SG phase operator is not Hermitian is nothing to be concerned about. However, as we now show, one can define an Hermitian phase operator, the measurement statistics of which converge, in an appropriate limit, to the phase distribution of (2.97) [13].

Consider the state $|\phi_0\rangle$ defined on a finite subspace of the oscillator Hilbert space by

$$|\phi_0\rangle = (s+1)^{-1/2} \sum_{n=1}^{s} e^{in\phi_0} |n\rangle \ . \tag{2.99}$$

It is easy to demonstrate that the states $|\phi\rangle$ with the values of ϕ differing from ϕ_0 by integer multiples of $2\pi/(s+1)$ are orthogonal. Explicitly, these states are

$$|\phi_m\rangle = \exp\left(i\frac{a^\dagger a\, m\, 2\pi}{s+1} \right) |\phi_0\rangle; \quad m = 0, 1, \ldots, s \ , \tag{2.100}$$

with

$$\phi_m = \phi_0 + \frac{2\pi m}{s+1} \ .$$

Thus $\phi_0 \le \phi_m < \phi_0 + 2\pi$. In fact, these states form a complete orthonormal set on the truncated $(s+1)$ dimensional Hilbert space. We now construct the *Pegg–Barnett* (PB) Hermitian phase operator

$$\phi = \sum_{m=1}^{s} \phi_m |\phi_m\rangle\langle\phi_m| \ . \tag{2.101}$$

For states restricted to the truncated Hilbert space the measurement statistics of ϕ are given by the discrete distribution

$$P_m = |\langle\phi_m|\psi\rangle_s|^2 \tag{2.102}$$

where $|\psi\rangle_s$ is any vector of the truncated space.

It would seem natural now to take the limit $s \to \infty$ and recover an Hermitian phase operator on the full Hilbert space. However, in this limit the PB phase operator does not converge to an Hermitian phase operator, but the distribution in (2.102) does converge to the SG phase distribution in (2.97). To see this, choose $\phi_0 = 0$.

Then

$$P_m = (s+1)^{-1} \left| \sum_{n=0}^{s} \exp\left(-i\frac{nm2\pi}{s+1} \right) \psi_n \right|^2 \tag{2.103}$$

where $\psi_n = \langle n|\psi\rangle_s$.

As ϕ_m are uniformly distributed over 2π we define the probability density by

$$P(\phi) = \lim_{s\to\infty} \left[\left(\frac{2\pi}{s+1} \right)^{-1} P_m \right] = \frac{1}{2\pi} \left| \sum_{n=1}^{\infty} e^{in\phi} \psi_n \right|^2 \tag{2.104}$$

where

$$\phi = \lim_{s\to\infty} \frac{2\pi m}{s+1} \ , \tag{2.105}$$

and ψ_n is the number state coefficient for any Hilbert space state. This convergence in distribution ensures that the moments of the PB Hermitian phase operator converge, as $s \to \infty$, to the moments of the phase probability density.

The phase distribution provides a useful insight into the structure of fluctuations in quantum states. For example, in the number state $|n\rangle$, the mean and variance of the phase distribution are given by

$$\langle \phi \rangle = \phi_0 + \pi \,, \tag{2.106}$$

and

$$V(\phi) = \frac{2}{3}\pi \,, \tag{2.107}$$

respectively. These results are characteristic of a state with random phase. In the case of a coherent state $|re^{i\theta}\rangle$ with $r \gg 1$, we find

$$\langle \phi \rangle = \phi \,, \tag{2.108}$$

$$V(\phi) = \frac{1}{4\bar{n}} \,, \tag{2.109}$$

where $\bar{n} = \langle a^\dagger a \rangle = r^2$ is the mean photon number. Not surprisingly a coherent state has well defined phase in the limit of large amplitude.

Exercises

2.1 If $|X_1\rangle$ is an eigenstate for the operator X_1 find $\langle X_1|\psi\rangle$ in the cases (a) $|\psi\rangle = |\alpha\rangle$; (b) $|\psi\rangle = |\alpha, r\rangle$.

2.2 Prove that if $|\psi\rangle$ is a minimum-uncertainty state for the operators X_1 and X_2, then $V(X_1, X_2) = 0$.

2.3 Show that the squeeze operator

$$S(r,\phi) = \exp\left[\frac{r}{2}\left(e^{-2i\phi}a^2 - e^{2i\phi}a^{\dagger 2}\right)\right]$$

may be put in the normally ordered form

$$S(r,\phi) = (\cosh r)^{-1/2} \exp\left(-\frac{\Gamma}{2}a^{\dagger 2}\right) \exp\left[-\ln(\cosh r)\, a^\dagger a\right] \exp\left(\frac{\Gamma^*}{2}a^2\right)$$

where $\Gamma = e^{2i\theta}\tanh r$.

2.4 Evaluate the mean and variance for the phase operator in the squeezed state $|\alpha, r\rangle$ with α real. Show that for $|r| \gg |\alpha|$ this state has either enhanced or diminished phase uncertainty compared to a coherent state.

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